Olympiad Corner

The 45th International Mathematical Olympiad took place on July 2004. Here are the problems.

Day 1 Time allowed: 4 hours 30 minutes.

Problem 1. Let $ABC$ be an acute-angled triangle with $AB \neq AC$. The circle with diameter $BC$ intersects the sides $AB$ and $AC$ at $M$ and $N$, respectively. Denote by $O$ the midpoint of the side $BC$. The bisectors of the angles $BAC$ and $MON$ intersect at $R$. Prove that the circumcircles of the triangles $BMR$ and $CNR$ have a common point lying on the side $BC$.

Problem 2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b)+P(b-c)+P(c-a) = 2P(a+b+c)$$

for all real numbers $a, b, c$ such that $ab + bc + ca = 0$.

Problem 3. Define a hook to be a figure made up of six unit squares as shown in the diagram

(continued on page 4)

IMO 2004

T. W. Leung

The 45th International Mathematical Olympiad (IMO) was held in Greece from July 4 to July 18. Since 1988, we have been participating in the Olympiads. This year our team was composed as follows.

Members

Cheung Yan Kuen (Hong Kong Chinese Women’s Club College)
Chung Tat Chi (Queen Elizabeth School)
Kwok Tsz Chiu (Yuen Long Merchant Association Secondary School)
Poony Ming Fung (STFA Leung Kau Kui College)
Tang Chiu Fai (HKTA Tang Hin Memorial Secondary School)
Wong Hon Yin (Queen’s College)
Cesar Jose Alaban (Deputy Leader)
Leung Tat Wing (Leader)

I arrived at Athens on July 6. After waiting for a couple of hours, leaders were then delivered to Delphi, a hilly town 170 km from the airport, corresponding to 3 more hours of journey. In these days the Greeks were still ecstatic about what they had achieved in the Euro 2004, and were busy preparing for the coming Olympic Games in August. Of course Greece is a small country full of legend and mythology. Throughout the trip, I also heard many times that they were the originators of democracy, their contribution in the development of human body and mind and their emphasis on fair play.

After receiving the short-listed problems leaders were busy studying them on the night of July 8. However obviously some leaders had strong opinions on the beauty and degree of difficulty of the problems, so selections of all six problems were done in one day. Several problems were not even discussed in details of their own merits.

The following days were spent on refining the wordings of the questions and translating the problems into different languages.

The opening ceremony was held on July 11. In the early afternoon we were delivered to Athens. After three hours of ceremony we were sent back to Delphi. By the time we were in Delphi it was already midnight. Leaders were not allowed to talk to students in the ceremony.

Contests were held in the next two days. The days following the contests were spent on coordination, i.e. leaders and coordinators discussed how many points should be awarded to the answers of the students. This year the coordinators were in general very careful. I heard several teams spent more than three hours to go over six questions. Luckily coordination was completed on the afternoon of July 15. The final Jury meeting was held that night. In the meeting the cut-off scores were decided, namely 32 points for gold, 24 for silver and 16 for bronze. Our team was therefore able to obtain two silver medals (Kwok and Chung) and two bronze medals (Tang and Cheung). Other members (Poony and Wong) both solved at least one problem completely, thus received honorable mention. Unofficially our team ranked 30 out of 85. The top five teams in order were respectively China, USA, Russia, Vietnam and Bulgaria.

In retrospect I felt that our team was good and balanced, none of the members was particularly weak. In one problem we were as good as any strong team. Every team members solved problem 4 completely. Should we did better in the geometry problems our rank would be much higher. Curiously geometry is in our formal school curriculum while number theory and combinatorics are not. In this Olympiad we had two geometry problems, but fittingly so, after all, it was Greece.
Extending an IMO Problem

Hà Duy Hung

Dept. of Math and Informatics
Hanoi Univ. of Education

In this brief note we give a generalization of a problem in the 41st
International Mathematical Olympiad held in Taejon, South Korea in 2000.

IMO 2000/5. Determine whether or not there exists a positive integer \( n \)
such that \( n + 1 \) is divisible by exactly 2000 different prime divisors, and
\( 2^n + 1 \) is divisible by \( n \).

The answer to the question is positive. This intriguing problem
made me recall a well-known theorem due to O. Reutter in [1] as follows.

**Theorem 1.** If \( a \) is a positive integer such that \( a + 1 \) is not a power of 2, then
\( a^x + 1 \) is divisible by \( n \) for infinitely many positive integers \( n \).

We frequently encounter the theorem in the case \( a = 2 \). The theorem and the IMO problem
prompted me to think of more general problem. Can we replace the number 2 in the IMO problem by other positive integers? The difficulty partly lies in
the fact that the two original problems are solved independently. After a long
time, I finally managed to prove a generalization as follows.

**Theorem 2.** Let \( s, a, b \) be given positive integers, such that \( a, b \) are relatively prime and \( a + b \) is not a power of 2. Then there exist infinitely many positive integers \( n \) such that

- \( n \) has exactly \( s \) different prime divisors; and
- \( a^x + b^y \) is divisible by \( n \).

We give a proof of Theorem 2 below. We shall make use of two familiar lemmas.

**Lemma 1.** Let \( n \) be an odd positive integer, and \( a, b \) be relative prime positive integers. Then
\[
\frac{a^x + b^y}{a + b} = n \]
is an odd integer \( \geq 1 \), equality if and only if \( n = 1 \) or \( a = b = 1 \).

The proof of Lemma 1 is simple and is left for the reader.

Also, we remind readers the usual notations \( r \mid s \) means \( s \) is divisible by \( r \) and
\( u \equiv v \pmod{m} \) means \( u - v \) is divisible by \( m \).

**Lemma 2.** Let \( a, b \) be distinct and relatively prime positive integers, and \( p \) an odd prime number which divides \( a + b \). Then for any non-negative integer \( k \),
\[
p^{k+1} \mid a^x + b^y ,
\]
where \( m = p^k \).

**Proof.** We prove the lemma by induction. It is clear that the lemma holds for \( k = 0 \). Suppose the lemma holds for some non-negative integer \( k \), and we proceed to the case \( k + 1 \).

Let \( x = a^x \) and \( y = b^y \). Since
\[
x^r + y^r = (x + y) \sum_{i=0}^{r-1} (-1)^i x^{r-1-i} y^i ,
\]
it suffices to show that the whole summation is divisible by \( p^k \). Since \( x = y \pmod{p^{k+1}} \), we have
\[
\sum_{i=0}^{r-1} (-1)^i x^{r-1-i} y^i \\
= \sum_{i=0}^{r-1} (-1)^i x^{r-1} y^i \\
= px^{r-1} \pmod{p^{k+1}}
\]
completing the proof.

In the rest of this note we shall complete the proof of Theorem 2.

**Proof of Theorem 2.** Without loss of generality, let \( a > b \). Since \( a + b \) is not a power of 2, it has an odd prime factor \( p \). For natural number \( k \), set
\[
x_k = a^x + b^y , \quad y_k = \frac{x_{k+1}}{x_k} .
\]
Then \( y_k \) is a positive integer and
\[
y_k = \sum_{i=0}^{r-1} (-1)^i (a^x)^{r-1-i} (b^y)^i \\
= \sum_{i=0}^{r-1} (-1)^i (a^x)^{r-1} \\
= px^{r-1} \pmod{p^{k+1}}
\]
which implies that \( \frac{y_k}{p} \) is a positive integer. Also, we have
\[
\frac{y_k}{p} = b^{y(p-1)} \pmod{\frac{x_k}{p}} ,
\]
so that
\[
\gcd \left( \frac{x_k}{p}, \frac{y_k}{p} \right) = 1
\]
for \( k = 1, 2, \ldots \) By Lemma 2, we also have
\[
\gcd \left( \frac{y_k}{p}, \frac{p^k}{p^k} \right) = 1
\]
for \( k = 1, 2, \ldots \). Moreover, we have \( x_k \geq p^k \). This leads us to
\[
y_k = b^{y(p-1)} + \sum_{i=0}^{r-1} [(a^x)^{r-1} (b^y)^{p-1-i}] \\
> b^{y(p-1)} \pmod{p^{k+1}} = x_k \geq p^{k+1}
\]
It follows that
\[
\frac{y_k}{p} \geq p^k > 1 .
\]
By Lemma 1, \( \frac{y_k}{p} \) is an odd positive integer, so we can choose an odd prime divisor \( q \) of \( \frac{y_k}{p} \).

We now have a sequence of odd prime numbers \( \{ q_i \}_{k=1}^{\infty} \) satisfying the
following properties

(i) \( \gcd (x_k, q_i) = 1 \)

(ii) \( \gcd (p, q_i) = 1 \)

(iii) \( q_i \mid x_{k+i} \)

(iv) \( x_k \mid x_{k+i} \).

We shall now show that the sequence \( \{ q_i \}_{k=1}^{\infty} \) consists of distinct prime numbers and is thus infinite. Indeed, if \( k_0 < k_1 \) are positive integers and \( q_{k_0} = q_{k_1} \), then
\[
q_{k_0} = q_{k_1} \mid x_{k_0+i} \mid \cdots \mid x_{k_1}
\]
by properties (iii) and (iv). But this contradicts property (i).

Next, set \( n_0 = p^0 q_1 \cdots q_{k_0-1} \) and \( n_{k+1} = pn_k \) for \( k = 0, 1, 2, \ldots \). It is evident that
\( \{ n_k \}_{k=0}^{\infty} \) is a strictly increasing sequence

(continued on page 4)
Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The submissions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to: Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is October 20, 2004.

Problem 206. (Due to Zdravko F. Stare, Vršac, Serbia and Montenegro) Prove that if \( a, b \) are the legs and \( c \) is the hypotenuse of a right triangle, then
\[
(a+b)\sqrt{a} + (a-b)\sqrt{b} < \sqrt{2^2 c} < \sqrt{2}\sqrt{2}\sqrt{c}.
\]

Problem 207. Let \( A = \{0, 1, 2, \ldots, 9\} \) and \( B_1, B_2, \ldots, B_i \) be nonempty subsets of \( A \) such that \( B_1 \) and \( B_2 \) have at most two common elements whenever \( i \neq j \). Find the maximum possible value of \( k \).

Problem 208. In \( \triangle ABC \), let \( AB > AC > BC \). Let \( D \) be a point on the minor arc \( BC \) of the circumcircle of \( \triangle ABC \). Let \( O \) be the circumcenter of \( \triangle ABC \). Let \( E, F \) be the intersection points of line \( AD \) with the perpendiculars from \( O \) to \( AB, AC \), respectively. Let \( P \) be the intersection of lines \( BE \) and \( CF \). If \( PB = PC + PO \), then \( \angle BAC \) is obtuse with proof.

Problem 209. Prove that there are infinitely many positive integers \( n \) such that \( 2^n + 2 \) is divisible by \( n \) and \( 2^n + 1 \) is divisible by \( n - 1 \).

Problem 210. Let \( a_1 = 1 \) and
\[
a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}
\]
for \( n = 1, 2, 3, \ldots \). Prove that for every integer \( n > 1 \),
\[
\frac{2}{\sqrt{a_n^2 - 2}}
\]
is an integer.

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Solutions
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Problem 201. (Due to Abderrahim OUARDINI, Talence, France) Find which nonright triangles \( \triangle ABC \) satisfy
\[
\tan A \tan B + \tan C > \tan A + \tan B + \tan C,
\]
where \( [t] \) denotes the greatest integer less than or equal to \( t \). Give a proof.

Solution. CHENg Hao (The Second High School Attached to Beijing Normal University), CHEUNG YUN Kuen (HKUST, Math, Year 1) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4). ACHILLEAS P. PORFYRIADIS (American College of Thessaloniki “Anatonia”, Thessaloniki, Greece),

From
\[
\tan C = \tan \left(180^\circ - A - B\right) = -\tan (A+B) = -(\tan A + \tan B)/(1-\tan A \tan B),
\]
we get
\[
\tan A + \tan B + \tan C = \tan A \tan B \tan C.
\]

Let \( x = \tan A, y = \tan B \) and \( z = \tan C \). If \( xyz \leq [x]+[y]+[z] \), then \( x+y+z \leq [x]+[y]+[z] \). As \([t] \leq t, x, y, z \) must be integers.

If triangle \( \triangle ABC \) is obtuse, say \( A > 90^\circ \), then \( x < 0 < 1 \leq y \leq z \). This implies \( 1 \leq yz = (x+y+z)/x = 1 + (y+z)/x < 1 \), a contradiction. If triangle \( \triangle ABC \) is acute, then we may assume \( 1 \leq x \leq y \leq z \). Now \( x+y+z = \frac{x+y+z}{3} \leq 3 \). Checking the cases \( x = 1, 2, 3 \), we see \( x+y+z = xyz \) can only happen when \( x=1, y=2, z=3 \). This corresponds to \( A = \tan^{-1} 1, B = \tan^{-1} 2 \) and \( C = \tan^{-1} 3 \). Reversing the steps, we see among nonright triangles, the inequality in the problem holds except only for triangles with angles equal \( 45^\circ \), \( \tan^{-1} 1 \), \( \tan^{-1} 2 \) and \( \tan^{-1} 3 \).

Problem 202. (Due to LUK Mee Lin, La Salle College) For triangle \( \triangle ABC \), let \( D, E, F \) be the midpoints of sides \( AB, BC, CA \), respectively. Determine which triangles \( \triangle ABC \) have the property that triangles \( \triangle ADF, \triangle BEF, \triangle CFE \) can be folded above the plane of triangle \( \triangle DEF \) to form a tetrahedron with \( \triangle ABC \) coincides with \( BD; \triangle DEF \) coincides with \( CE; \triangle ABC \) with \( AF \).

Solution. CHENg Hao (The Second High School Attached to Beijing Normal University), CHEUNG YUN Kuen (HKUST, Math, Year 1) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

Observe that \( \angle ADF = \angle BAC \) and \( \angle ADF = \angle ABC \). If \( \angle ADF = \angle BAC \), then \( \angle ADF = \angle BAC \) coincides with \( \angle BAC \). In order for \( \triangle ABC \) to coincide with \( BD \) in folding, we need to have \( \angle ADF + \angle ADF + \angle ADF = \angle BAC \). So we need \( \angle BAC = \angle ABC = \angle BCA \). Similarly, for \( \triangle DEF \) to coincide with \( \triangle BAC \). If \( \angle DEF = \angle BAC \) and \( \angle DEF = \angle BAC \) coincides with \( \angle BAC \). So no angle of \( \triangle BAC \) is 90 or more. Therefore, \( \triangle ABC \) is acute.

Conversely, if \( \triangle ABC \) is acute, then reversing the steps, we can see that the required tetrahedron can be obtained.

Problem 203. (Due to José Luis DIAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain) Let \( a, b, c \) be real numbers such that \( a+b+c \neq 0 \). Prove that the equation \( (a+b+c)x^2 + 2(ab+bc+ca)x + 3abc = 0 \) has only real roots.

Solution. CHAN Pak Woon (Wah Yan College, Kowloon, Form 6), CHENg Hao (The Second High School Attached to Beijing Normal University), CHEUNG Hoi Kit (SHK Lam Kau Mow Secondary School, Form 7), CHEUNG YUN Kuen (HKUST, Math, Year 1), MURRAY Klamkin (University of Alberta, Edmonton, Canada), ACHILLEAS P. PORFYRIADIS (American College of Thessaloniki “Anatonia”, Thessaloniki, Greece) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

The quadratic has real roots if and only if its discriminant
\[
D = 4(ab+bc+ca)^2 - 12(ab+bc+ca) - 2\sqrt{[a] + [b] + [c]} = 4(ab+bc+ca)^2 - 2[ab+bc+ca] = 4(ab+bc+ca)^2 - 2[ab+bc+ca] \]
is nonnegative, which is clear.

Other commended solvers: Jason CHENG Hoi Sing (SHK Lam Kau Mow Secondary School, Form 7), POON Ho Yin (Munsang College (Hong Kong Island), Form 4) and Anderson TORRES (Universidade de Sao Paulo – Campus Sao Carlos).

Problem 204. Let \( n \) be an integer with \( n > 4 \). Prove that for every \( n \) distinct integers taken from 1, 2, ..., 2n, there always exist two numbers whose least common multiple is at most \( 3n + 6 \).

Solution. CHENg Hao (The Second High School Attached to Beijing Normal University), CHEUNG YUN Kuen (HKUST, Math, Year 1) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

Let \( S \) be the set of \( n \) integers taken and \( k \) be the minimum of these integers. If \( k \leq n \), then either \( 2k \) is also in \( S \) or \( 2k \) is not in \( S \). In the former case, \( \text{lcm}(k, 2k) = 2k \leq 2n \leq 3n+6 \). In the latter case, we replace \( k \) in \( S \) by \( 2k \). Note this will not
decrease the least common multiple of any pair of numbers. So if the new $S$ satisfies the problem, then the original $S$ will also satisfy the problem. As we repeat this, the new minimum will increase strictly so that we eventually reach either $k$ or $2k$ both in $S$, in which case we are done, or the new $S$ will consist of $n+1, n+2, \ldots, 2n$. So we need to consider the latter case only.

If $n > 4$ is even, then $3(n+2)/2$ is an integer at most $2n$ and $\text{lcm}(n+2, 3(n+2)/2) = 3n/6$. If $n > 4$ is odd, then $3(n+1)/2$ is an integer at most $2n$ and $\text{lcm}(n+1, 3(n+1)/2) = 3n/3$.

**Problem 205.** (Due to HA Duy Hung, Hanoi University of Education, Vietnam) Let $a, n$ be integers, both greater than 1, such that $a^2 - 1$ is divisible by $n$. Prove that the greatest common divisor (or highest common factor) of $a - 1$ and $n$ is greater than 1.

**Solution.** CHENG Hao (The Second High School Attached to Beijing Normal University), CHEUNG Yun Kuen (HKUST, Math, Year 1) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

Let $p$ be the smallest prime divisor of $n$. Then $a^2 - 1$ is divisible by $p$ so that $a^2 - 1 \equiv 0 \pmod{p}$. In particular, $a$ is not divisible by $p$. Then, by Fermat’s little theorem, $a^{n-1} = 1 \pmod{p}$.

Let $d$ be the smallest positive integer such that $a^d - 1 \equiv 0 \pmod{p}$. Dividing $n$ by $d$, we get $n = dq + r$ for some integers $q, r$ with $0 \leq r < d$. Then $a^r \equiv (a^d)^q a^r \equiv a^r \equiv 1 \pmod{p}$. By the definition of $d$, we get $r = 0$. Then $n$ is divisible by $d$. Similarly, dividing $p - 1$ by $d$, we see $a^d - 1 \equiv 0 \pmod{p}$ implies $p - 1$ is divisible by $d$. Hence, $\gcd(n, p - 1)$ is divisible by $d$. Since $p$ is the smallest prime dividing $n$, we must have $\gcd(n, p - 1) = 1$. So $d = 1$. By the definition of $d$, we get $a - 1$ is divisible by $p$. Therefore, $\gcd(a - 1, n) \geq p > 1$.

**Olympiad Corner**

(continued from page 1)

![rectangle]

or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that
- the rectangle is covered without gaps and without overlaps;
- no part of a hook covers area outside the rectangle.

Day 2 Time allowed: 4 hours 30 minutes.

**Problem 4.** Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > \left( t_1 + t_2 + \cdots + t_n \right) \times \left( \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n} \right).$$

Show that $t_i$, $t_j$, $t_k$ are side lengths of a triangle for all $i, j, k$ with $1 \leq i < j < k \leq n$.

**Problem 5.** In a convex quadrilateral $ABCD$ the diagonal $BD$ bisects neither the angle $ABC$ nor the angle $CDA$. The point $P$ lies inside $ABCD$ and satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

**Problem 6.** We call a positive integer alternating if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers $n$ such that $n$ has a multiple which is alternating.

**Extending an IMO Problem**

(continued from page 2)

of positive integers and each term of the sequence has exactly $s$ distinct prime divisors.

It remains to show that

$$n_k \mid a^{k^s} + b^{k^s}$$

for $k = 0, 1, 2, \ldots$. Note that for odd positive integers $m, n$ with $m \mid n$, we have $a^m + b^m \mid a^n + b^n$. By property (iii), we have, for $0 \leq k < s$,

$$q_i \mid x_{i+1} \mid x_i \mid a^{k^i} + b^{k^i} \mid a^{k^{i+1}} + b^{k^{i+1}}$$

for $j = 0, 1, 2, \ldots$. Now it suffices to show that

$$p^{k+s} \mid a^{k^s} + b^{k^s}$$

for $k = 0, 1, 2, \ldots$. But this follows easily from Lemma 2 since

$$p^{k+s} \mid x_{i+s} \mid a^{k^s} + b^{k^s}.$$ This completes the proof of Theorem 2.

**References:**


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2004 Hong Kong team to IMO: From left to right, Cheung Yun Kuen, Poon Ming Fung, Tang Chiu Fai, Cesar Jose Alaban (Deputy Leader), Leung Tat Wing (Leader), Chung Tat Chi, Kwok Tsz Chiu & Wong Hon Yin.
solution. Since $OM = ON$, the bisector of $\angle MON$ coincides with the perpendicular bisector of $MN$. So in $\triangle AMN$ the bisector of $\angle MAN$ and the perpendicular bisector of the side $MN$ meet at $R$. Then, as is well known, $R$ lies on the circumcircle of the triangle. (One needs to note that $AM \not= AN$; indeed, $\angle LAN = \angle IC$ and $\angle LAN = \angle LB$ from the cyclic quadrilateral $BCNM$, and $LB + IC$ by hypothesis.)

Let the bisector of $\angle ABC$ meet $BC$ at $L$. It is easy to see that $L$ is a common point of the circles $(AMN)$ and $(ONR)$. This means to show that the quadrilaterals $BCLR$ and $CLR$ are cyclic, which is equivalent to $\angle ANM = \angle ABC$ and $\angle LAC = \angle LCB$. But $\angle LAC = \angle LAN$ and $\angle LCB = \angle LAM$, as mentioned above, so the question reduces to $\angle LAC = \angle LAM$, $\angle LCR = \angle LAM$.

These angle equalities follow from the cyclic quadrilateral $AMRN$, and the solution is complete.

Solution. For every real $x$ the triple $(a, b, c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. For this triple the equation gives

$$P(3x) + P(5x) + P(\frac{-5x}{2}) = 2P(\frac{7x}{2})$$

for all $x \in \mathbb{R}$.

If $P(x) = \sum a_n x^n$, this implies

$$(3^2 + 5^2 + (-8)^2 + 2 \cdot 7) a_n = 0$$

for all $i = 0, 1, 2, \ldots$. The expression in the brackets is negative for odd $i$ and positive for $i = 0$ and for all even $i \geq 6$. Only for $i = 2$ and $i = 4$ this expression is 0. It follows that $P(x) = ax^2 + bx^4$, with $a, b \in \mathbb{R}$. As before, only the verification remains.

Solution. Consider a covering of an $m \times n$ rectangle satisfying the conditions. Fix any hook $A$, there is a unique hook $B$ covering the "inside" square of $A$ with one of its "endmost" squares. In turn, the "inside" square of $B$ must be covered by an "endmost" square of $A$. Thus, in a tiling, all hooks are interlocked into pairs. There are only two possibilities to place $B$ so that it does not overlap with $A$ and no gap occurs. In one case, $A$ and $B$ form a $3 \times 4$ rectangle; in the other, their union is an octagonal shape, with side-lengths 3, 2, 1, 2, 1, 2, 1, 2. So an $m \times n$ rectangle can be covered with hooks if and only if it can be covered with the $12$-square tiles shown above. Suppose that such a tiling exists; then $mn$ is divisible by 12. We now show that one of $m$ and $n$ is divisible by 4.

Assume on the contrary that this is not the case; then $m$ and $n$ are both even, hence $mn$ is divisible by 4. Imagine the rectangle divided into unit squares, with the rows and columns numbered labeled 1, $\ldots$, $m$ and 1, $\ldots$, $n$. Write 1 in the square $(i, j)$ if exactly one of $i$ and $j$ is divisible by 4, and 2, if both $i$ and $j$ are divisible by 4. Since the number of squares in each row and column is even, the sum of all numbers written is even. Now, it is easy to check that a $3 \times 4$ rectangle always covers numbers with sum 3 or 1, and the other $12$-square shape always covers numbers with sum 5 or 7. Consequently, the total number of $12$-square shapes is even. But then $mn$ is divisible by 24, and hence by 4, contrary to the assumption that $m$ and $n$ are not divisible by 4.

Notice also that neither $m$ nor $n$ can be 1, 2, or 5 (any attempt to place tiles along a side of length 1, 2, or 5 fails). We infer that if a tiling is possible, then one of $m$ and $n$ is divisible by 3, one is divisible by 4, and $mn \not= (1, 2, 5)$.

Conversely, if these conditions are satisfied, the tiling is possible (using only $3 \times 4$ rectangles at that). This is immediate if 3 divides $m$ and 4 divides $n$ (or vice versa). Let $m$ be divisible by 12 and $n \not= (1, 2, 5)$ (or vice versa). Then $n$ can be represented as the sum of several 3's and several 4's. Hence the rectangle can be partitioned into $m \times 3$ and $m \times 4$ rectangles, which are easy to cover, only with $3 \times 4$ tiles again.

Solution. By symmetry, it suffices to show that $t_1 < t_2 + t_3$. One has

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \left( \frac{t_i}{t_j} \right) = n + \sum_{i<j} \left( \frac{t_i}{t_j} \right)$$

$$= n + \left( \frac{1}{t_i} \right) \left( \sum_{i=1}^{n} t_i \right) + \left( \frac{1}{t_j} \right) \left( \sum_{i=1}^{n} t_i \right) + \sum_{i<j} \left( \frac{t_i}{t_j} \right)$$

By the AM-GM inequality,

$$\frac{1}{t_i} + \frac{1}{t_j} \geq \frac{2}{\sqrt{t_i t_j}}$$

and $i, j \geq 2$ for all $i, j$. Thus, setting $a = t_1/\sqrt{t_2 t_3} > 0$ and using the hypothesis, we obtain

$$n^2 + 1 > \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i + 2 \left( \frac{n}{2} - \frac{2}{a} \right) = 2a + \frac{2}{a} + n^2 - 4$$

Hence $2a + \frac{2}{a} - 5 < 0$, which implies $\frac{1}{2} < a = \frac{t_1}{\sqrt{t_2 t_3}} < 2$. So $t_1 < 2\sqrt{t_2 t_3}$, and one more application of the AM-GM inequality yields $t_1 < 2\sqrt{t_2 t_3} \leq t_2 + t_3$, as needed.

Solution. Since $P$ is interior to $ABCD$, it follows that $\angle DDA < \angle DBC$ if and only if $\angle DAB < \angle DCB$. So one may assume without loss of generality that $P$ lies in the triangles $ACD$ and $BCD$.

Assume that the quadrilateral $ABCD$ is cyclic. Let the lines $BP$ and $DP$ meet $AC$ at $K$ and $L$, respectively. It follows from the given equalities and $\angle ABC = \angle ADB$, $\angle DAB = \angle ACD$ that the triangles $ABK$, $BKC$ and $CKB$ are similar. This implies $\angle DPK = \angle KPI$, so $PK = PL$.

The triangles $ADL$ and $BDC$ are also similar, hence

$$\frac{AD}{CD} = \frac{KE}{CD}$$

yielding $AL = EC$. Combined with the conclusions above, this implies that the triangles $ALP$ and $KCP$ are congruent. Hence $AP = CP$.

Conversely, assume that $AP = CP$. Let the circumcircle of the triangle $BCP$ meet the lines $CD$ and $DP$ again at $X$ and $Y$, respectively. The triangles $ADB$ and $PDX$ are similar, implying that $APD$ and $BDX$ are also similar. Therefore

$$\frac{BX}{AP} = \frac{BD}{AD} = \frac{XD}{PD}$$

Moreover, the triangles $DPX$ and $DXY$ are similar, which gives

$$\frac{YX}{OP} = \frac{XD}{PD}$$

Since $AP = CP$, it follows from (1) and (2) that $BX = YX$. Hence

$$\angle DCB = \angle XBY = \angle AYX = \angle AYP + \angle PXD = \angle ADB + \angle DAB = 180^\circ - \angle BAD$$

The above equality means that $ABCD$ is a cyclic quadrilateral.
Solution. A positive integer \( n \) has an alternative multiple if and only if \( n \) is not divisible by 20.

If \( n \) is divisible by 20 then its last two decimal digits are even, hence no alternative multiple exists. The remaining \( n \) have alternative multiples.

We consider separately the powers of 2 and the numbers of the form \( 2 \cdot 5^k \). In the sequel, \( u^k \) means that \( u^k \) is the highest power of \( u \) dividing \( n \).

Lemma 1. Each power of 2 has an alternative multiple with an even number of digits.

Proof of Lemma 1. It suffices to construct an infinite sequence \( \{a_n\}_{n=1}^{\infty} \) of decimal digits such that

\[
a_n \equiv n + 1 \pmod{2}; \quad 2^{2n-1} \mid a_{2n-1} \ldots a_{1}; \quad 2^{2n} \mid a_{2n}a_{2n-1} \ldots a_{1},
\]

for each \( n \). Start with \( a_1 = 2, a_2 = 7 \). If the sequence is constructed up to \( a_{2n} \), set \( a_{2n+1} = 4 \). Then \( a_{2n+1} \) is even and

\[
2^{2n+1} \mid a_{2n+1}a_{2n-1} \ldots a_{1} = 4 \cdot 10^m \cdot a_{2n}a_{2n-1} \ldots a_{1},
\]

because \( 2^{2n+1} \mid a_{2n}a_{2n-1} \ldots a_{1} \) by the inductive hypothesis, and \( 2^{2n} \mid 4 \cdot 10^m \). Denote \( a_{2n+1} \ldots a_{1} = 2^{2n+1}A \), with \( A \) odd. Now, \( a_{2n+2} \) must be odd and such that

\[
2^{2n+3} \mid a_{2n+2}a_{2n+1} \ldots a_{1} = a_{2n+2}10^{2n+1} + a_{2n+1}a_{2n} \ldots a_{1} = 2^{2n+1}[a_{2n+3}5^{n+1} + A],
\]

which holds whenever \( 5a_{2n+2} + A \equiv 4 \pmod{8} \). The solutions of the last congruence are odd, since \( A \) is odd. In addition, any solution \( a_{2n+2} \) can be chosen from \( \{0, 1, \ldots, 7\} \). The construction is complete.

Lemma 2. Each number of the form \( 2 \cdot 5^k, n = 1, 2, \ldots \), has an alternative multiple with an even number of digits.

Proof of Lemma 2. We construct an infinite sequence \( \{b_n\}_{n=1}^{\infty} \) of decimal digits such that

\[
b_n \equiv n + 1 \pmod{2}; \quad \text{and} \quad 2 \cdot 5^k \text{ divides } b_{n+1} \ldots b_1,
\]

for each \( n \). One can start with \( b_1 = 0, b_2 = 5 \). Suppose that \( b_1, \ldots, b_n \) are constructed, and let \( b_1 \ldots b_\ell = 5^kB \), where \( \ell \geq n \) and \( B \) is not divisible by 5. The next digit \( b_{n+1} \) must be such that \( b_{n+1} \equiv n + 2 \pmod{2} \) and \( 5^k \) divides

\[
b_{n+1}b_n \ldots b_{1} = b_{n+1}10^n + b_n \ldots b_{1} = 5^k[b_{n+1}2^{n} + 5^{\ell-n}B].
\]

The latter is true whenever \( b_{n+1}2^n + D \) is divisible by 5. Now, the system of simultaneous congruences \( b_{n+1} \equiv n + 2 \pmod{2}, b_{n+1}2^n + B \equiv 0 \pmod{5} \) has a solution by the Chinese remainder theorem, since \( 2^n \) and \( 5 \) are coprime. Also, a solution \( b_{n+1} \) can be chosen in \( \{0, 1, \ldots, 9\} \), as needed.

We pass on to the case of a general \( n = 2^a5^b k \), where \( k \) is coprime to 10. If \( n \) is not divisible by 20 then \( 2^a5^b \) is a power of 2, a power of 5, or a number of the form \( 2 \cdot 5^k \). By Lemmas 1 and 2, in all cases \( 2^a5^b \) has an even alternative multiple \( M \) with an even number \( 2m \) of digits. Clearly, all integers of the form \( MM \ldots M \) are alternative multiples of \( 2^a5^b \), too. We prove that some of them is a multiple of \( n = 2^a5^b k \). Consider the numbers

\[
C_k = 1 + 10^m + \cdots + 10^{m(t-1)}, \quad t = 1, 2, \ldots, k + 1.
\]

Some of them, \( C_0 \) and \( C_k \) with \( t < t_1 \), are congruent modulo \( k \) by the pigeonhole principle. Hence \( k \) divides their difference \( C_k - C_0 = C_{k-t_1} \cdot 10^{m(t_1)} \). And because \( k \) is coprime to 10, it follows that \( k \) divides \( C_{t_1} \). Now it is straightforward that \( C_{t_1} \times M \), a number of the form \( MM \ldots M \), is an alternative multiple of \( n \).