Olympiad Corner

The Czech-Slovak-Polish Match this year took place in Bilovec on June 21-22, 2004. Here are the problems.

Problem 1. Show that real numbers $p$, $q$, $r$ satisfy the condition

$$p^3(q - r)^2 + 2p^2(q + r) + 1 = p^4$$

if and only if the quadratic equations

$$x^2 + px + q = 0$$

and

$$y^2 - py + r = 0$$

have real roots (not necessarily distinct) which can be labeled by $x_1$, $x_2$ and $y_1$, $y_2$, respectively, in such way that the equality $x_1y_1 + x_2y_2 = 1$ holds.

Problem 2. Show that for each natural number $k$ there exist at most finitely many triples of mutually distinct primes $p$, $q$, $r$ for which the number $qr - k$ is a multiple of $p$, the number $pr - k$ is a multiple of $q$, and the number $pq - k$ is a multiple of $r$.

Problem 3. In the interior of a cyclic quadrilateral $ABCD$, a point $P$ is given such that $|\measuredangle BPC| = |\measuredangle BAP| = |\measuredangle DPC|$. Denote by $E$, $F$ and $G$ the feet of the perpendiculars from the point $P$ to the lines $AB$, $AD$ and $DC$, respectively. Show that the triangles $FEG$ and $PBC$ are similar.

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Homothety

Kin Y. Li

A geometric transformation of the plane is a function that sends every point on the plane to a point in the same plane. Here we will like to discuss one type of geometric transformations, called homothety, which can be used to solve quite a few geometry problems in some international math competitions.

A homothety with center $O$ and ratio $k$ is a function that sends every point $X$ on the plane to the point $X'$ such that

$$OX' = kOX.$$ 

So if $|k| > 1$, then the homothety is a magnification with center $O$. If $|k| < 1$, it is a reduction with center $O$. A homothety sends a figure to a similar figure.

Example 1. (1981 IMO) Three congruent circles have a common point $O$ and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenters and the circumcenters of the triangle and the point $O$ are collinear.

Solution. Consider the figure shown. Let $A'$, $B'$, $C'$ be the centers of the circles. Since the radii are the same, so $A'B'$ is parallel to $AB$, $B'C'$ is parallel to $BC$, $C'A'$ is parallel to $CA$. Since $AA'$, $BB'$, $CC'$ are the bisector of $\angle A$, $\angle B$, $\angle C$ respectively, they concur at the incenter $I$ of $\triangle ABC$. Note $O$ is the circumcenter of $\triangle A'B'C'$ as it is equidistant from $A'$, $B'$, $C'$. Then the homothety with center $I$ sending $\triangle A'B'C'$ to $\triangle ABC$ will send $O$ to the circumcenter $P$ of $\triangle ABC$. Therefore, $I, O, P$ are collinear.

Example 2. (1982 IMO) A non-isosceles triangle $A_1A_2A_3$ is given with sides $a_1$, $a_2$, $a_3$, $a_i$ is the side opposite $A_i$. For all $i = 1, 2, 3$, $M_i$ is the midpoint of side $a_i$, and $T_i$ is the point where the incirlce touches side $a_i$. Denote by $S_i$ the reflection of $T_i$ in the interior biserctor of angle $A_i$.

Prove that the lines $M_1S_1$, $M_2S_2$ and $M_3S_3$ are concurrent.
Solution. Let $I$ be the incenter of $\triangle A_1A_2A_3$. Let $B_1$, $B_2$, $B_3$ be the points where the internal angle bisectors of $\angle A_1$, $\angle A_2$, $\angle A_3$ meet $a_1$, $a_2$, $a_3$ respectively. We will show $S_3$ is parallel to $M_1M_2$. With respect to $A_1B_1$, the reflection of $T_1$ is $S_1$ and the reflection of $T_2$ is $T_3$. So $\angle T_1S_1 = \angle T_3T_2$. With respect to $A_2B_2$, the reflection of $T_2$ is $S_2$ and the reflection of $T_1$ is $S_3$. So $\angle T_3S_3 = \angle T_1T_2$. Then $\angle T_3S_3 = \angle T_1S_1$. Since $IT_1$ is perpendicular to $A_1A_2$, we get $S_3S_1$ is parallel to $A_1A_2$. Since $A_1A_2$ is parallel to $M_2M_3$, we get $S_3S_1$ is parallel to $M_2M_3$. Similarly, $S_2S_2$ is parallel to $M_1M_2$ and $S_3S_3$ is parallel to $M_1M_3$.

Now the circumcircle of $\triangle S_1S_2S_3$ is the incircle of $\triangle A_1A_2A_3$ and the circumcircle of $\triangle M_1M_2M_3$ is the nine point circle of $\triangle A_1A_2A_3$. Since $A_1A_2A_3$ is not equilateral, these circles have different radii. Hence $\triangle S_1S_2S_3$ is not congruent to $\triangle M_1M_2M_3$ and there is a homothety sending $\triangle S_1S_2S_3$ to $\triangle M_1M_2M_3$. Then $M_1S_1$, $M_2S_2$ and $M_3S_3$ concur at the center of the homothety.

Example 4. (1983 IMO) Let $A$ be one of the two distinct points of intersection of two unequal coplanar circles $C_1$ and $C_2$ with centers $O_1$ and $O_2$ respectively. One of the common tangents to the circles touches $C_1$ at $P_1$ and $C_2$ at $P_2$, while the other touches $C_1$ at $Q_1$ and $C_2$ at $Q_2$. Let $M_1$ be the midpoint of $P_1Q_1$ and $M_2$ be the midpoint of $P_2Q_2$. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

Solution. By symmetry, lines $O_2O_1$, $P_1P_3$, $Q_3Q_1$, concur at a point $O$. Consider the homothety with center $O$ which sends $C_1$ to $C_2$. Let $OA$ meet $C_2$ at $B$, then $A$ is the image of $B$ under the homothety. Since $\triangle BM_1O_1$ is sent to $\triangle AM_2O_2$, so $\angle M_1BO_1 = \angle M_2AO_2$.

Now $\triangle OP_1O_1$ similar to $\triangle OM_1P_1$ implies $OO_1/OP_1 = OP_1/OM_1$. Then

$$OO_1 \cdot OM_1 = OP_1^2 = OA \cdot OB,$$

which implies points $A$, $B$, $M_1$, $O$ are concyclic. Then $\angle M_1BO_1 = \angle M_2AO_1$. Hence $\angle M_1AO_1 = \angle M_2AO_2$. Adding $\angle O_1AM_2$ to both sides, we have $\angle O_1AO_2 = \angle M_1AM_2$.

Example 5. (1992 IMO) In the plane let $C$ be a circle, $L$ a line tangent to the circle $C$, and $M$ a point on $L$. Find the locus of all points $P$ with the following property: there exist two points $Q, R$ on $L$ such that $M$ is the midpoint of $QR$ and $C$ is the inscribed circle of $\triangle PQR$.

Solution. Let $L$ be the tangent to $C$ at $S$. Let $T$ be the reflection of $S$ with respect to $M$. Let $U$ be the point on $C$ diametrically opposite $S$. Take a point $P$ on the locus. The homothety with center $M$ that sends $G_1$ to $G_2$ will send $C$ to some point $A'$ and line $EF$ to the tangent line of $\Gamma$ at $A'$. Since lines $EF$ and $L$ are parallel, $A'$ must be the midpoint of arc $FA'E$. Then $\angle A'EC = \angle A'FC = \angle A'ME$. So $\triangle A'EC$ is similar to $\triangle A'ME$. Then the power of $A'$ with respect to $\Gamma$ is $A'C' \cdot A'M = A'E^2$. Similar, the power of $A'$ with respect to $\Gamma_2$ is $A'F^2$. Since $A'E = A'F$, $A'$ has the same power with respect to $\Gamma_1$ and $\Gamma_2$. So $A'$ is on the radical axis $AB$. Hence, $A' = A$. Then $C' = C$ and $C$ is on $EF$.

Similarly, the other common tangent to $\Gamma_1$ and $\Gamma_2$ passes through $D$. Let $O_2$ be the center of $\Gamma_2$. By symmetry with respect to $O_2O_1$, we see that $O_2$ is the midpoint of arc $CO_2D$. Then

$$\angle DO_1C = \angle DCO_2 = \angle FCO_2.$$

This implies $O_2$ is on the angle bisector of $\angle FCD$. Since $CF$ is tangent to $\Gamma_2$, therefore $CD$ is tangent to $\Gamma_2$.

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is January 20, 2005.

Problem 211. For every $a, b, c, d$ in $[1,2]$, prove that

$$
\frac{a+b}c + \frac{c+d}d \leq 4 \frac{a+c}{b+d}.
$$

(Source: 32nd Ukrainian Math Olympiad)

Problem 212. Find the largest positive integer $N$ such that if $S$ is any set of 21 points on a circle $C$, then there exist $N$ arcs of $C$ whose endpoints lie in $S$ and each of the arcs has measure not exceeding $120^\circ$.

Problem 213. Prove that the set of all positive integers can be partitioned into positive integers can be partitioned into $k$ subsets such that if $i \neq j$, then at least two of them belong to the same subset.

Problem 214. Let the inscribed circle of triangle $ABC$ be tangent to sides $AB$, $BC$ at $E$ and $F$ respectively. Let the angle bisector of $\angle CAB$ intersect segment $EF$ at $K$. Prove that $\angle EKA$ is a right angle.

Problem 215. Given a 8x8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by $21 \times 1$ rectangles.

************** Solutions

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Problem 206. (Due to Zdravko F. Starc, Vraca, Serbia and Montenegro) Prove that if $a, b$ are the legs and $c$ is the hypotenuse of a right triangle, then

$$(a+b)\sqrt{a} + (a-b)\sqrt{b} < \sqrt{2\sqrt{2}c}\sqrt{c}.$$ 

Solution. Cheng HAO (The Second High School Attached to Beijing Normal University). HUI Jack (Queen’s College, Form 5). D. Kipp JOHNSON (Valley Catholic School, Teacher, Beaverton, Oregon, USA). POON Ming Fung (STFA Leung Kau Kui College, Form 7). Achilleas P. PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece), Problem Group Discussion Euler-Teorema (Fortaleza, Brazil), Anna Ying PUN (STFA Leung Kau Kui College, Form 6). TO Ping Leung (St. Peter’s Secondary School) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

By Pythagoras’ theorem,

$$a + b \leq \sqrt{(a + b)^2 + (a - b)^2} = \sqrt{2c}.$$

Equality if and only if $a = b$. By the Cauchy-Schwarz inequality,

$$\left(\frac{a + b}{\sqrt{a}} + \frac{a - b}{\sqrt{b}}\right)^2 \leq \frac{(a + b)^2}{a} + \frac{(a - b)^2}{b} \leq \frac{2\sqrt{2c}}{\sqrt{2c}}.$$

For equality to hold throughout, we need $a + b : a - b = \sqrt{a} : \sqrt{b} = 1 : 1$, which is not possible for legs of a triangle. So we must have strict inequality.

Other commended solvers: CHAN Wai Hung (Carmel Divine Grace Foundation Secondary School, Form 6), LI Sai Ki (Carmel Divine Grace Foundation Secondary School, Form 6), LING Shu Dung, Anna Ying PUN (STFA Leung Kau Kui College, Form 6) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

Problem 208. In $\triangle ABC$, $AB > AC > BC$. Let $D$ be a point on the minor arc $BC$ of the circumcircle of $\triangle ABC$. Let $O$ be the circumcenter of $\triangle ABC$. Let $E, F$ be the intersection points of line $AD$ with the perpendiculars from $O$ to $AB, AC$, respectively. Let $P$ be the intersection of lines $BE$ and $CF$. If $PB = PC + PO$, then $\angle BAC$ has proof with proof.

Solution. Achilles P. PORFYRIADIS (American College of Thessaloniki “Anatolia”, Thessaloniki, Greece). Problem Group Discussion Euler-Teorema (Fortaleza, Brazil) and Anna Ying PUN (STFA Leung Kau Kui College, Form 6).

Since $E$ is on the perpendicular bisector of chord $AB$ and $F$ is on the perpendicular bisector of chord $AC$, $AE = BE$ and $AF = CF$. Applying exterior angle theorem,

$$\angle BPC = \angle AEP + \angle CFD = 2(\angle BAD + \angle CAD) = 2 \angle BAC = \angle BOC.$$ 

Hence, $B, C, P, O$ are concyclic. By Ptolemy’s theorem,

$$PB \cdot OC = PC \cdot OB + PO \cdot BC.$$ 

Then $(PB - PC) \cdot OC = PO \cdot BC$. Since $PB - PC = PO$, we get $OC = BC$ and so $\triangle OBC$ is equilateral. Then

$$\angle BAC = \frac{1}{2} \angle BOC = 30^\circ.$$ 

Other commended solvers: Cheng HAO (The Second High School Attached to Beijing Normal University), HUI Jack (Queen’s College, Form 5), POON Ming Fung (STFA Leung Kau Kui College, Form 7), TONG Yiu Wai.
Problem 209. Prove that there are infinitely many positive integers $n$ such that $2^n + 2$ is divisible by $n$ and $2^{n+1}$ is divisible by $n - 1$.

Solution. D. Kipp Johnson (Valley Catholic School, Teacher, Beaverton, Oregon, USA), POON Ming Fung (St. Francis Leung Kau Kui College, Form 7) and Problem Group Discussion Euler-Theorema (Fortaleza, Brazil).

As $2^2 + 2 = 6$ is divisible by 2 and $2^4 + 1 = 5$ is divisible by 1, $n = 2$ is one such number.

Next, suppose $2^n + 2$ is divisible by $n$ and $2^{n+1}$ is another such number. We will prove $N = 2^n + 2$ is another such number. Since $N - 1 = 2^n + 1 = (n - 1)k$ is odd, so $k$ is odd and $n$ is even. Since $N = 2^n + 2 = 2(2^{n-1} + 1) = nm$ and $n$ is even, so $m$ must be odd. Recall the factorization

\[ x + 1 = (x + 1)(x^2 - x^4 + \ldots + 1) \]

for odd positive integer $i$. Since $k$ is odd, $2^n + 2 = 2(2^{n-1} + 1) = (0^{n-1}1)$ is divisible by $2(2^{n-1} + 1) = 2^n + 2 = N$ using the factorization above. Since $m$ is odd, $2^n + 1 = 2^{n+1} + 1$ is divisible by $2^n + 1 = N - 1$. Hence, $N$ is also such a number. As $N > n$, there will be infinitely many such numbers.

Problem 210. Let $a_1 = 1$ and

\[ a_{n+1} = \frac{a_n^2 + 1}{2a_n} \]

for $n = 1, 2, 3, \ldots$ Prove that for every integer $n > 1$,

\[ \frac{2}{\sqrt{a_n^2 - 2}} \]

is an integer.

Solution. G.R.A. 20 Problem Group (Roma, Italy), HUDREÁ Mihail (High School “Tiberiu Popoviciu” Cluj-Napoca Romania), Problem Group Discussion Euler–Teorema (Fortaleza, Brazil), TO Ping Leung (St. Peter’s Secondary School) and YIM Wing Yin (South Tuen Mun Government Secondary School, Form 4).

Note $a_n = p_n/q_n$, where $p_1 = q_1 = 1$, $p_{n+1} = p_n^2 + 2q_n^2$, $q_{n+1} = 2p_nq_n$, for $n = 1, 2, 3, \ldots$. Then

\[ \frac{2}{\sqrt{a_n^2 - 2}} = \frac{2q_n}{\sqrt{p_n^2 - 2q_n^2}}. \]

It suffices to show by mathematical induction that $p_n^2 - 2q_n^2 = 1$ for $n > 1$. We have $p_2^2 - 2 = 3^2 - 2 = 3 = 1$. Assuming case $n$ is true, we get

\[ p_{n+1}^2 - 2q_{n+1}^2 = (p_n^2 + 2q_n^2)^2 - 2(2p_nq_n) = (p_n^2 - 2q_n^2)^2 = 1. \]

Olympiad Corner (continued from page 1)

Problem 4. Solve the system of equations

\[ \frac{1}{xy} = \frac{x}{z} + 1, \quad \frac{1}{yz} = \frac{y}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1 \]

in the domain of real numbers.

Problem 5. In the interiors of the sides $AB$, $BC$ and $CA$ of a given triangle $ABC$, points $K$, $L$ and $M$, respectively, are given such that

\[ \frac{AK}{KB} = \frac{BL}{LC} = \frac{CM}{MA}. \]

Show that the triangles $ABC$ and $KLM$ have a common orthocenter if and only if the triangle $ABC$ is equilateral.

Problem 6. On the table there are $k$ heaps of $1, 2, \ldots, k$ stones, where $k \geq 3$. In the first step, we choose any three of the heaps on the table, merge them into a single new heap, and remove 1 stone (throw it away from the table) from this new heap. In the second step, we again merge some three of the heaps together into a single new heap, and then remove 2 stones from this new heap. In general, in the $i$-th step we choose any three of the heaps, which contain more than $i$ stones when combined, we merge them into a single new heap, then remove $i$ stones from this new heap. Assume that after a number of steps, there is a single heap left on the table, containing $p$ stones. Show that the number $p$ is a perfect square if and only if the numbers $2k+2$ and $3k+1$ are perfect squares. Further, find the least number $k$ for which $p$ is a perfect square.

Homothety (continued from page 2)

Example 7. (2000APMO) Let $ABC$ be a triangle. Let $M$ and $N$ be the points in which the median and the angle bisector, respectively at $A$ meet the side $BC$. Let $Q$ and $P$ be the points in which the perpendicular at $N$ to $NA$ meets $MA$ and $BA$ respectively and $O$ the point in which the perpendicular at $P$ to $BA$ meets $AN$ produced.

Prove that $PQ$ is perpendicular to $BC$. 

Solution (due to Bobby Poon). The case $AB = AC$ is clear.

Without loss of generality, we may assume $AB > AC$. Let $AN$ intersect the circumcircle of $\triangle ABC$ at $D$. Then

\[ \angle DBC = \angle DAC = \frac{1}{2} \angle BAC \]

\[ = \angle DAB = \angle DCB. \]

So $DB = DC$ and $MD$ is perpendicular to $BC$.

Consider the homothety with center $A$ that sends $\triangle DBC$ to $\triangle OBC'$. Then $OB' = OC'$ and $BC$ is parallel to $BC'$. Let $B'C'$ intersect $PN$ at $K$. Then

\[ \angle OB'K = \angle DBC = \angle DAB \]

\[ = 90^\circ - \angle AOP = \angle OPK. \]

So points $P$, $B'$, $O$, $K$ are concyclic. Hence $\angle B'KO = \angle B'PO = 90^\circ$ and $B'K = C'K$. Since $BC$ is parallel to $B'C'$, this implies $K$ is on $AM$. Hence, $K = Q$. Since $\angle B'KO = 90^\circ$ and $BC$ is parallel to $B'C'$, we get $QO$ is perpendicular to $BC$. 

Problem 1

First part. Assume that the equations (2) have real roots satisfying \( x_1 y_1 - x_2 y_2 = 1 \). By the familiar formula, the roots of the quadratic equations are given by

\[
\begin{align*}
  x_{1,2} &= \frac{-p \pm K}{2} \\
  y_{1,2} &= \frac{p \pm L}{2},
\end{align*}
\]

where the real numbers \( K, L \) satisfy \( K^2 = p^2 - 4q \) and \( L^2 = p^2 - 4r \) (we choose the signs of \( K, L \) in accordance with the labelling of the roots). Thus

\[
1 = x_1 y_1 - x_2 y_2 = \frac{(-p + K)(p + L) - (-p - K)(p - L)}{4} = \frac{p(K - L)}{2},
\]

whence \( p \neq 0 \) and \( K - L = 2/p \). Substituting this into the equality

\[
(K + L)(K - L) = K'^2 - L^2 = (p^2 - 4q) - (p^2 - 4r) = 4(r - q),
\]

yields \( K + L = 2p(r - q) \). From these values of \( K + L \) and \( K - L \) we obtain \( K = 1/p - p(q - r) \) and \( L = -p(q - r) + 1 \). Comparing this with the equality \( K'^2 = p^2 - 4q \), an easy manipulation leads to the desired equation (1).

Second part. Assume that (1) holds. Then clearly \( p \neq 0 \). The equation (1) can be rewritten in either of the following two forms,

\[
p(q - r)^2 + 2p^2(q - r) + 1 = p^2 - 4p^2r \quad \text{and} \quad p^2(r - q)^2 + 2p^2(r - q) + 1 = p^4 - 4p^2q.
\]

Upon dividing by \( p^2 \) we find that the discriminants of the equations (2) are equal to

\[
p^2 - 4q = \left( \frac{p^2(r - q) + 1}{p} \right)^2 \quad \text{and} \quad p^2 - 4r = \left( \frac{p^2(q - r) + 1}{p} \right)^2;
\]

hence, they are nonnegative and the (real) roots of (2) have the form (3), where

\[
K = \frac{p^2(r - q) + 1}{p} \quad \text{and} \quad L = \frac{p^2(q - r) + 1}{p}.
\]

The signs of the numbers \( K \) and \( L \) have been chosen so that (see First Part)

\[
x_1 y_1 - x_2 y_2 = \frac{K}{2} \cdot \left( \frac{p^2(r - q) + 1}{p} + \frac{p^2(q - r) + 1}{p} \right) = 1.
\]

Problem 2

(Mutually distinct) primes \( p, q, r \) satisfy the desired conditions if and only if the number \( pq + pr + qr - k \) is divisible by each of the primes \( p, q, r, \) that is, by the product \( pqr \). The equality \( pq + pr + qr - k = n \cdot pqr \), for a suitable integer \( n \), can be rewritten as \( k = pq + pr + qr - n \cdot pqr \). If \( n \leq 0 \), then the last equality implies that \( \max(p, q, r, k) \leq k \). However, then each of the primes \( p, q, r \) is less than or equal to \( k/2 \) (and there is only a finite number of such triples). If \( n \geq 1 \), then we get the estimate \( k \leq pq + pr + qr - pqr \). Let us show that the last expression is negative (contradicting the fact that \( k > 0 \)) unless the triple in question is \( (p, q, r) = (2, 3, 5) \). We can assume that \( 2 \leq p < \sqrt{3} < r \) and \( r > 7 \). Then \( pq \geq 2 \cdot 3 - 6 \) and the inequality \( (p - 2)(q - 2) \geq 0 \) implies that \( p + q \leq 3pq + 2 \), hence

\[
pq + pr + qr - pqr = (p + q)r + pq + pqr \leq 3pq + 2r + pqr
\]

\[= 2r - pq(\frac{3}{2}r - 1) \leq 2r - 6(\frac{3}{2}r - 1) = 6 - r < 0.
\]

Let \( k \) be the circumcircle of the quadrangle \( ABCD \) and \( k_1, k_2 \) the circumcircles of the triangles \( PAB \) and \( PCD \), respectively. In the interior of the angle \( BPC \), consider the half-line \( PT \) such that \( |\angle BPT| = |\angle BAP| \). Then the hypothesis on \( P \) implies that (Fig. 1)

\[
|\angle TCP| = |\angle PBC| - |\angle BPT| = |\angle BPC| - |\angle DAP| = |\angle PDC|.
\]

Thus \( PT \) is the common interior tangent of the circles \( k_1 \) and \( k_2 \).
Since the quadrangle $ABCD$ is cyclic, $\angle QDA = \angle QBC$. From the relations (1) and (5) we thus get
\[ |\angle FEG| = |\angle QBC| - |\angle PBA| = |\angle PBC|. \]

Using finally the relations (3) and (6) we see that the triangles $EFG$ and $PBC$ are similar (as they have two congruent angles).

An analogous argument can be used when the point $Q$ is located on the half-line $AB$ beyond the points $B$. If the lines $AB$ and $CD$ are parallel, then $ABCD$ is an equilateral trapezoid, with bases $AB$ and $CD$. Since the points $E$, $F$, $G$ are collinear and the common interior tangent of the circles $k_1$ and $k_2$ is parallel to both lines $AB$ and $CD$, the triangles $APD$ and $BPC$ are congruent. The similarity of the triangles $EFG$ and $APD$ thus implies also the similarity of the triangles $EFG$ and $BPC$. This completes the proof.

**Problem 4**

From the form of the equations it is immediate that $xyz \neq 0$. Two of the numbers $x$, $y$, $z$ must have to be of the same sign; then the right-hand side of the equation where the ratio of these two numbers occurs is positive, hence so must be the corresponding left-hand side, which implies that the third of the numbers $x$, $y$, $z$ must also have the same sign as the first and the second. Thus either $x, y, z > 0$, or $x, y, z < 0$. Let us consider only the former case (the latter can be reduced to it by passing from the solution $(x, y, z)$ to the solution $(-x, -y, -z)$). Multiply the first two equations of the system by the expression $xyz$ and then subtract them; this gives, upon a small manipulation, $x - y = y(z^2 - yz)$. If a triple $(x, y, z)$ is a solution, then so are also the triples $(y, x, z)$ and $(z, y, x)$; thus we may assume that $x = \max(x, y, z)$. Then $y = x - y > 0$ and $z = z - y > 0$ (remember that $x, y, z > 0$), so the equality $x - y = y(z^2 - yz)$, together with the condition $y > 0$, implies that $x = y = z$. The system thus reduces to the single equation $x^2 = 1 + 1$, which has a (unique) positive root $x = \sqrt{2}/2$.

Conclusion. The system has exactly two solutions, $x = y = z = \pm\sqrt{2}/2$.

**Problem 5**

A point $V$ of the plane containing a triangle $ABC$ is its orthocenter if and only if, at the same time, $AV \perp BC$ and $BV \perp AC$; that is, $AV \cdot BC = 0$ and $BV \cdot AC = 0$. Substituting $BC = BV - CV$ and $AC = AV - CV$, an easy manipulation leads to the equivalent condition in the form of the equality of the scalar products
\[ AV \cdot BV = AV \cdot CV = BV \cdot CV. \]

Our goal is thus to find out when the system (1) is satisfied together with the analogous system
\[ KV \cdot LV = KV \cdot MV = LV \cdot MV, \]
expressing the fact that the point $V$ is the orthocenter of the triangle $KLM$. We now express the vectors from (2) as linear combinations of the vectors from (1). By hypothesis, there exists a number $p$, $0 < p < 1$, for which
\[ AK = pAB, \quad BL = pBC, \quad CM = pCA. \]

Substituting into the first equality $AK = AV - KV$ and $AB = AV - BV$, we get after a small manipulation the first of the following three equalities
\[ KV = (1 - p)AV + pBV, \quad LV = (1 - p)BV + pCV, \quad MV = (1 - p)CV + pAV; \]
the other two can be derived similarly. Taking products, we get
\[ KV \cdot LV = (1 - p)^2 AV \cdot BV + p(1 - p)AV \cdot CV + p(1 - p)BV^2 = -(1 - p)s + p(1 - p)BV^2, \]
where $s$ denotes the common value of the products from (1). Similarly,
\[ KV \cdot MV = (1 - p)s + p(1 - p)AV^2 \quad \text{and} \quad LV \cdot MV = (1 - p)s + p(1 - p)BV^2. \]

We see that the system (2) is equivalent to the system of equalities
\[ p(1 - p)AV^2 = -(1 - p)s + p(1 - p)BV^2, \]
which, in view of the condition $p(1 - p) \neq 0$, is fulfilled if and only if $|AV| = |BV| = |CV|$. The last condition means that the orthocenter $V$ of the triangle $ABC$ coincides with its circumcenter. This happens if and only if the triangle $ABC$ is equilateral.