Olympiad Corner

Below is the Czech-Polish-Slovak Match held in Zward on June 20-21, 2005.

Problem 1. Let \( n \) be a given positive integer. Solve the system of equations

\[
x_1 + x_2 + x_3 + \cdots + x_n = n, \\
x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n(n+1)/2
\]

in the set of nonnegative real numbers \( x_1, x_2, \ldots, x_n \).

Problem 2. Let a convex quadrilateral \( ABCD \) be inscribed in a circle with center \( O \) and circumscribed to a circle with center \( I \) and let its diagonals \( AC \) and \( BD \) meet at a point \( P \). Prove that the points \( O, I \) and \( P \) are collinear.

Problem 3. Determine all integers \( n \geq 3 \) such that the polynomial \( W(x) = x^n - 3x^{n-1} + 2x^n + 6 \) can be expressed as a product of two polynomials with positive degrees and integer coefficients.

Problem 4. We distribute \( n \geq 1 \) labelled balls among nine persons \( A, B, C, D, E, F, G, H, I \). Determine in how many ways

(continued on page 4)

Using Tangent Lines to Prove Inequalities

Kin-Yin Li

For students who know calculus, sometimes they become frustrated in solving inequality problems when they do not see any way of using calculus. Below we will give some examples, where finding the equation of a tangent line is the critical step to solving the problems.

**Example 1.** Let \( a, b, c, d \) be positive real numbers such that \( a + b + c + d = 1 \). Prove that

\[
6(a^3+b^3+c^3+d^3) \geq (a^2+b^2+c^2+d^2)^2 + 1/8.
\]

**Solution.** We have \( 0 < a, b, c, d < 1 \). Let \( f(x) = 6x^3 - x^2 \). (Note: Since there is equality when \( a = b = c = d = 1/4 \), we consider the graph of \( f(x) \) and its tangent line at \( x = 1/4 \). By a simple sketch, it seems the tangent line is below the graph of \( f(x) \) on the interval \((0,1)\). Now the equation of the tangent line at \( x = 1/4 \) is \( y = (5x - 1)/8 \). So we claim that for \( 0 < x < 1 \) we have

\[
f(x) - f(1/4) \geq (x - 1/4)[f(1/4) - f(1/4)] + 1/8 = 1/8.
\]

**Example 2.** (2003 USA Math Olympiad) Let \( a, b, c \) be positive real numbers. Prove that

\[
(2a+b+c)^2 + (2b+c+a)^2 + (2c+a+b)^2 \leq 8(a+b+c)^2.
\]

**Solution.** Setting \( a' = a(a+b+c), b' = b(a+b+c), c' = c(a+b+c) \) if necessary, we may assume \( 0 < a, b, c < 1 \) and \( a+b+c = 1 \). Then the first term on the left side of the inequality is equal to

\[
f(a) = \frac{(a+1)^2}{2a^2 + (1-a)^2} = \frac{a^2 + 2a + 1}{3a^2 - 2a + 1}.
\]

(Note: When \( a = b = c = 1/3 \), there is equality. A simple sketch of \( f(x) \) on \([0,1]\) shows the curve is below the tangent line at \( x = 1/3 \), which has the equation \( y = (12x + 4)/3 \).) So we claim that

\[
a^2 + 2a + 1 \leq \frac{12a + 4}{3a^2 - 2a + 1}
\]

for \( 0 < a < 1 \). Multiplying out, we see this is equivalent to

\[
s(3a^2 - 2a + 1)^2 \geq 0
\]

for \( 0 < a < 1 \). (Note: Since the curve and the line intersect at \( a = 1/3 \), we expect \( 3a - 1 \) is a factor.) Indeed, \( 36a^3 - 15a^2 - 2a + 1 \geq 0 \) for \( 0 < a < 1 \). Finally adding the similar inequality for \( b \) and \( c \), we get the desired inequality.

The next example looks like the last example. However, it is much more sophisticated, especially without using tangent lines. The solution below is due to Titu Andreescu and Gabriel Dospinescu.

**Example 3.** (1997 Japanese Math Olympiad) Let \( a, b, c \) be positive real numbers. Prove that

\[
(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \geq \frac{3}{5}(b^3+c^3+a^3).
\]

**Solution.** As in the last example, we may assume \( 0 < a < 1 \) and \( a + b + c = 1 \). Then the first term on the left becomes

\[
(1-2a)^2 + \frac{2}{2-a^2 + 2} + \frac{2}{2-a^2 + 2} - \frac{2}{a+b+c}.
\]

Next, let \( a_1 = 1-2a, x_2 = 1-2b, x_3 = 1-2c \). Then \( x_1 + x_2 + x_3 = 1, \) but \( x_1 < x_2, x_3 < 1 \). In terms of \( x_1, x_2, x_3 \), the desired inequality is

\[
f(x_1) + f(x_2) + f(x_3) \leq 27/10.
\]

(Note: As in the last example, we consider the equation of the tangent line to \( f(x) = 1/(1+x^2) \) at \( x = 1/3 \), which is \( y = 27(-x + 2)/50 \). So we claim that \( f(x) \leq 27(-x + 2)/50 \) for \(-1 < x < 1 \). This is equivalent to \( (3x - 1)(4 - 3x) \geq 0 \). Hence the claim is true for \(-1 < x < 1 \). Then \( f(x_1) + f(x_2) + f(x_3) \leq 27/10 \) and the desired inequality follows.)
Schur’s Inequality
Kin Yin Li

Sometimes in proving an inequality, we do not see any easy way. It will be good to know some brute force methods in such situation. In this article, we introduce a simple inequality that turns out to be very critical in proving inequalities by brute force.

Schur’s Inequality. For any \( x, y, z \geq 0 \) and \( r > 0 \),
\[
x'(x-y)(x-z) + y'(y-x)(y-z) + z'(z-x)(z-y) \\
+ z'(z-x)(z-y) \geq 0.
\]
Equality holds if and only if \( x = y = z \) or two of \( x, y, z \) are equal and the third is zero.

Proof. Observe that the inequality is symmetric in \( x, y, z \). So without loss of generality, we may assume \( x \geq y \geq z \). Then \( x'(x-y)(x-z) \geq y'(x-y)(y-z) \) so that the sum of the first two terms is nonnegative. As the third term is also nonnegative, so the sum of all three terms is nonnegative. In case \( x \geq y \geq z \), equality holds if and only if \( x = y \) first and \( z \) equals to them or zero.

In using the Schur’s inequality, we often expand out expressions. So to simplify writing, we introduce the symmetric sum notation \( \sum_{\text{sym}} f(x,y,z) \) to denote the sum of the six terms \( f(x,y), f(y,z), f(z,x), f(x,z), f(x,y), f(z,y) \) and \( f(x,y,z) \). In particular,
\[
\sum_{\text{sym}} x^3 = 2x^3 + 2y^3 + 2z^3,
\]
\[
\sum_{\text{sym}} x^3y = x^3y + x^3z + y^3z + y^3x + z^3x + z^3y
\]
and
\[
\sum_{\text{sym}} xyz = 6xyz.
\]
Similarly, for a function of \( n \) variables, the symmetric sum is the sum of all \( n! \) terms, where we take all possible permutations of the \( n \) variables.

The \( r = 1 \) case of Schur’s inequality is
\[
x(x-y)(x-z) + y(y-x)(y-z) + z(z-x)(z-y) = x^3 + y^3 + z^3 - (x^2y + x^2z + y^2z + y^2x + z^2x + z^2y) + 3xyz \geq 0.
\]
In symmetric sum notation, it is
\[
\sum_{\text{sym}} (x^3 - 2x^2y + xyz) \geq 0.
\]
By expanding both sides and rearranging terms, each of the following inequalities is equivalent to the \( r = 1 \) case of Schur’s inequality. These are common disguises.

\( a) \ x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + xz(x+z), \)
\( b) \ xy \geq (x+y-z)(y+z-x)(z+x-y), \)
\( c) \ 4(x+y+z)(x^2+y^2+z^2) + 2xyz \leq (x+y+z)^2 + 9xyz. \)

Example 1. (2000 IMO) Let \( a, b, c \) be positive real numbers such that \( abc = 1 \). Prove that
\[
(a - 1 + \frac{1}{b})(b - 1 + \frac{1}{c})(c - 1 + \frac{1}{a}) \leq 1.
\]
Solution. Let \( a = x/y, b = y/z \) and \( c = z/x \). Substituting these into the desired inequality, we get
\[
\frac{x-y+z}{y} \left( \frac{y-z+x}{z} \left( \frac{z-x+y}{x} \right) \right) \leq 1,
\]
which is disguise \( b) \) of the \( r = 1 \) case of Schur’s inequality.

Example 2. (1984 IMO) Prove that
\[
0 \leq yz + zx + xy - 2xyz \leq 7/27,
\]
where \( x, y, z \) are nonnegative real numbers such that \( x + y + z = 1 \).

Solution. In Schur’s inequality, all terms are of the same degree. So we first change the desired inequality to one where all terms are of the same degree. Since \( x + y + z = 1 \), the desired inequality is the same as
\[
0 \leq (x + y + z)(yz + zx + xy) - 2xyz \leq \frac{7(x+y+z)^3}{27}.
\]
Expanding the middle expression, we get
\[
xyz + \sum_{\text{sym}} x^3y, \]
which is clearly nonnegative and the left inequality is proved. Expanding the rightmost expression and subtracting the middle expression, we get
\[
\frac{7}{54} \sum_{\text{sym}} (x^3 - \frac{12}{7} x^2y + \frac{5}{7} xyz).
\]

By Schur’s inequality, we have
\[
\sum_{\text{sym}} (x^3 - 2x^2y + xyz) \geq 0.
\]
By the AM-GM inequality, we have
\[
\sum_{\text{sym}} x^3y \geq 6(x^4y^2z^3)^{1/6} = \sum_{\text{sym}} xyz,
\]
which is the same as
\[
\sum_{\text{sym}} (x^2y - xyz) \geq 0.
\]

Multiplying (3) by 2/7 and adding it to (2), we see the symmetric sum in (1) is nonnegative. So the right inequality is proved.

Example 3. (2004 APMO) Prove that
\[
(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)
\]
for any positive real numbers \( a, b, c \).

Solution. Expanding and expressing in symmetric sum notation, the desired inequality is
\[
(ab)^2 + \sum_{\text{sym}} (a^2b^2 + 2ab) + 8 \geq \frac{9}{2} \sum_{\text{sym}} ab.
\]
As \( a^2 + b^2 + 2ab \), we get \( \sum_{\text{sym}} a^2 \geq \sum_{\text{sym}} ab \).

As \( a^2b^2 + 1 \geq 2ab \), we get
\[
\sum_{\text{sym}} a^2b^2 + 6 \geq 2 \sum_{\text{sym}} ab.
\]
Using these, the problem is reduced to showing
\[
(ab)^2 + 2 \geq \sum_{\text{sym}} (ab - \frac{1}{2} a^2).
\]
To prove this, we apply the AM-GM inequality twice and disguise c) of the \( r = 1 \) case of Schur’s inequality as follow:
\[
(ab)^2 + 2 \geq 3(ab)^{2/3}
\]
\[
\geq 9abc/(a+b+c)
\]
\[
\geq 4(ab + bc + ca) - (a + b + c)^2
\]
\[
= 2(ab + bc + ca) - (a^2 + b^2 + c^2)
\]
\[
= \sum_{\text{sym}} (ab - \frac{1}{2} a^2).
\]

Example 4. (2000 USA Team Selection Test) Prove that for any positive real numbers \( a, b, c \), the following inequality holds
\[
a + b + c \leq \frac{3}{2} \sqrt{abc}
\]
\[
\leq \max \{ (\sqrt{a} - \sqrt{b})^2, (\sqrt{b} - \sqrt{c})^2, (\sqrt{c} - \sqrt{a})^2 \}.
\]

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Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver’s name, home (or email) address and school affiliation. Please send submissions to Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong. The deadline for submitting solutions is February 12, 2006.

Problem 241. Determine the smallest possible value of
\[
S = a_1 a_2 a_3 + b_1 b_2 b_3 + c_1 c_2 c_3,
\]
if \(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\) is a permutation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. (Source: 2002 Belarussian Math Olympiad)

Problem 242. Prove that for every positive integer \(n\), 7 is a divisor of \(3^n + n^3\) if and only if 7 is a divisor of \(3^n + 1\). (Source: 1995 Bulgarian Winter Math Competition)

Problem 243. Let \(R'\) be the set of all positive real numbers. Prove that there is no function \(f : R' \rightarrow R'\) such that
\[
(f(x))^2 \geq f(x + y)f(x + y)
\]
for arbitrary positive real numbers \(x\) and \(y\). (Source: 1998 Bulgarian Math Olympiad)

Problem 244. An infinite set \(S\) of coplanar points is given, such that every three of them are not collinear and every two of them are not nearer than 1 cm from each other. Does there exist any division of \(S\) into two disjoint infinite subsets \(R\) and \(B\) such that inside every triangle with vertices in \(R\) is at least one point of \(B\) and inside every triangle with vertices in \(B\) is at least one point of \(R\)? Give a proof to your answer. (Source: 2002 Albanian Math Olympiad)

Problem 245. \(ABCD\) is a concave quadrilateral such that \(\triangle BAD = \angle ABC = \angle CDA = 45^\circ\). Prove that \(AC = BD\).

Problem 237. Determine (with proof) all polynomials \(p\) with real coefficients such that \(p(x)p(x + 1) = p(x^2)\) holds for every real number \(x\). (Source: 2000 Bulgarian Math Olympiad)

Solution. YEUNG Wai Kit (STFA Leung Kui Kui College, Form 5)

Let \(p(x)\) be such a polynomial. In case \(p(x)\) is a constant polynomial, \(p(x)\) must be 0 or 1. For the case \(p(x)\) is nonconstant, let \(r\) be a root of \(p(x)\). Then setting \(x = r\) and \(x + 1 = r\) in the equation, we see \(r^2\) and \((r - 1)^2\) are also roots of \(p(x)\). Also, \(r^2\) is a root implies \((r^2 - 1)^2\) is also a root. If 0 < \(|r| < 1\) or \(|r| > 1\), then \(p(x)\) will have infinitely many roots \(r, r^2, r^3, \ldots\), a contradiction. So \(|r| = 0\) or 1 for every root \(r\).

Without loss of generality, we can assume the center of the pizza is at the origin \(O\) and one of the cuts is parallel to the \(x\)-axis (that is, \(BC\) is parallel to \(AD\) in the picture). Let \(P\) be the intersection of the \(x\)-axis and the \(60^\circ\)-cut. Let \(A'D'\) be parallel to the \(120^\circ\)-cut \(B'C'\). Let \(P''\) be the intersection of \(BC\) and \(A'D'\). Then \(\Delta PP''P''\) is equilateral. This implies the belts \(ABCD\) and \(A'B'C'D'\) have equal width. Since \(AD > A'D'\), the area of the belt \(ABCD\) is greater than the area of the belt \(A'B'C'D'\). Now when the area of the belt \(ABCD\) is subtracted from the total area of the shaded regions and the area of \(A'B'C'D'\) is then added,

we get exactly half the area of the pizza. Therefore, the claim follows.

Problem 236. Alice and Barbara order a pizza. They choose an arbitrary point \(P\), different from the center of the pizza and they do three straight cuts through \(P\), which pairwise intersect at \(60^\circ\) and divide the pizza into 6 pieces. The center of the pizza is not on the cuts. Alice chooses one piece and then the pieces are taken clockwise by Barbara, Alice, Barbara, Alice and Barbara. Which piece should Alice choose first in order to get more pizza than Barbara? (Source: 2002 Slovenian National Math Olympiad)

Solution. (Official Solution)

Let Alice choose the piece that contains the center of the pizza first. We claim that the total area of the shaded regions below is greater than half of the area of the pizza.

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Problem 239. (Due to José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain) In any acute triangle $ABC$, prove that

$$\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{C-A}{2}\right) \leq \frac{\sqrt{2}}{2} \left(\frac{a+b+c}{\sqrt{a^2+b^2+c^2}} + \frac{b+c+a}{\sqrt{b^2+c^2+a^2}} + \frac{c+a+b}{\sqrt{c^2+a^2+b^2}}\right)$$

**Solution.** (Proposer’s Solution)

By cosine law and the AM-GM inequality,

$$1 - 2\sin^2\frac{A}{2} = \cos A = \frac{b^2 + c^2 - a^2}{2bc} \geq \frac{b^2 + c^2 - a^2}{b^2 + c^2} = 1 - \frac{a^2}{b^2 + c^2}.$$

So $\sin \frac{A}{2} \leq \frac{a}{\sqrt{2(b^2 + c^2)}}$.

By sine law and $\cos(A/2) = \sin((B+C)/2)$, we get

$$\frac{a}{b+c} = \sin \frac{A}{2} \sin B + \sin C = \frac{2\sin(A/2) \cos(A/2)}{2\sin(B/2) \cos(C/2)} \cos(B-C/2) = \frac{\sin(A/2)}{2\sin(B/2) \cos(C/2)}.$$

Then

$$\cos(B-C/2) = \frac{b+c}{a} \sin \frac{A}{2} \leq \frac{\sqrt{2}}{2} \frac{b+c}{\sqrt{b^2 + c^2}}.$$

Adding two similar inequalities, we get the desired inequality.

Commended solvers: Anna Ying PUN (STFA Leung Kau Kui College, Form 7) and YEUNG Wai Kit (STFA Leung Kau Kui College, Form 5).

Problem 240. Nine judges independently award the ranks of 1 to 20 to twenty figure-skaters, with no ties. No two of the rankings awarded to any figure-skater differ by more than 3. The nine rankings of each are added. What is the maximum of the lowest of the sums? Prove your answer is correct. (Source: 1968 All Soviet Union Math Competitions)

**Solution.** WONG Kwok Kit (Carmel Divine Grace Foundation Secondary School, Form 7) and YEUNG Wai Kit (STFA Leung Kau Kui College, Form 5).

Suppose the 9 first places go to the same figure skater. Then 9 is the lowest sum.

Now 24 is possible if skaters $A, B, C$ all received 3 first, 3 third and 3 fourth places; skater $D$ received 5 second and 4 fifth places; skater $E$ received 4 second and 5 fourth places; and skater $F$ received 6 sixth places. Therefore, 24 is the answer.

Olympiad Corner

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Problem 4. (Cont.) it is possible to distribute the balls under the condition that $A$ gets the same number of balls as the persons $B$, $C$, $D$ and $E$ together.

Problem 5. Let $ABCD$ be a given convex quadrilateral. Determine the locus of the point $P$ lying inside the quadrilateral $ABCD$ and satisfying

$$[PAB] [PCD] = [PBC] [PDA],$$

where $[XYZ]$ denotes the area of triangle $XYZ$.

Problem 6. Determine all pairs of integers $(x, y)$ satisfying the equation

$$y(x + y) = x^3 - 7x^2 + 11x - 3.$$
1. Let \( n \) be a given positive integer. Solve the system of equations
\[
x_1 + x_2^2 + x_3^3 + \ldots + x_n^n = n,
\]
\[
x_1 + 2x_2 + 3x_3 + \ldots + nx_n = \frac{n(n+1)}{2}
\]
in the set of nonnegative real numbers \( x_1, x_2, \ldots, x_n \).

Solution. Suppose \( x_1, x_2, \ldots, x_n \) satisfy the equations above. Then we have
\[
0 = x_1 + x_2^2 + x_3^3 + \ldots + x_n^n - n - (x_1 + 2x_2 + 3x_3 + \ldots + nx_n - \frac{n(n+1)}{2}) = (x_1 - 2x_2 + 2 - 1) + (x_2 - 3x_3 + 3 - 1) + \ldots + (x_n^n - nx_n + n - 1).
\]
However, the expressions in the brackets are nonnegative. Indeed, for \( k > 2 \) and \( x > 0 \) we have, by the AM-GM inequality,
\[
x_k^k + k - 1 = x_k^k + 1 + \ldots + 1 \geq k \cdot \sqrt[k]{x_k^k} = kx
\]
and the equality holds if and only if \( x = 1 \). Therefore we have \( x_2 = x_3 = \ldots = x_n = 1 \) and, by the first equation, \( x_1 = 1 \).

2. Let a convex quadrilateral \( ABCD \) be inscribed in a circle with center \( O \) and circumscribed to a circle with center \( I \), and let its diagonals \( AC \) and \( BD \) meet at a point \( P \). Prove that the points \( O, I, \) and \( P \) are collinear.

Solution. Assume that the lines \( AI, BI, CI, DI \) meet the circumcircle of the quadrilateral \( ABCD \) at \( E, F, G, H \), respectively. Since the lines \( AI, BI, CI, DI \) are the bisectors of the respective angles of the quadrilateral \( ABCD \), the lines \( EG \) and \( FH \) are the diameters of the circumcircle of \( ABCD \). Thus \( EG \) and \( FH \) meet at \( O \).

Denote by \( X \) the point of intersection of \( EB \) and \( CH \). Using Pascal's theorem for the hexagon \( ACNBDE \) we see that \( P, X \) and \( I \) are collinear. Once again Pascal's theorem applied to the hexagon \( GCHFB \) yields that \( O, X \) and \( I \) are collinear. Thus \( O, I \) and \( P \) are collinear, as desired.

3. Determine all integers \( n \geq 3 \) such that the polynomial
\[
W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6
\]
can be expressed as a product of two polynomials with positive degrees and integer coefficients.

Solution. We check that for \( n = 3 \)
\[
x^3 - 3x^2 + 2x + 6 = (x+1)(x^2 - 4x + 6)
\]
Suppose that for \( n = 4 \) we have
\[
x^4 - 3x^3 + 2x^2 + 6 = (x^2 + ax + b)(x^2 + cx + d).
\]
Then, comparing the coefficients we obtain
\[
a + c = -3, \quad ac + b + d = 2, \quad bd = 6.
\]
The first equation implies that \( a \) and \( c \) are of different parity and therefore, by the second equality, \( b \) and \( d \) are of the same parity. This contradicts the third equation.

Hence we may assume that \( n \geq 5 \). Suppose that we have
\[
W(x) = P(x)Q(x),
\]
where
\[
P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,
\]
\[
Q(x) = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_1x + b_0,
\]
and \( a_k = b_{n-k} = \pm 1 \). With no loss of generality we may assume that \( k = \lfloor \frac{n}{2} \rfloor \leq n - 2 \) (because \( n \geq 5 \)). Comparing the coefficients of both sides of (1) we obtain the following system of equations:
\[
a_0b_0 = 6,
\]
\[
a_0b_1 + a_1b_0 = 0,
\]
\[
a_0b_2 + a_1b_1 + a_2b_0 = 0,
\]
\[
\vdots
\]
\[
a_0b_k + a_1b_{k-1} + \ldots + a_kb_0 = 0.
\]
Now we can easily prove by induction that \( a_0 \) divides \( a_1, a_2, \ldots, a_k \). Indeed, having established this fact for \( a_1, a_2, \ldots, a_l \), we write
\[
0 = a_0(a_1b_{l+1} + a_1b_l + \ldots + a_lb_1 + a_0b_0) =
\]
\[
= a_0b_{l+1} + a_0a_1b_l + \ldots + a_0a_kb_0 + 6a_0b_1 + 6a_0b_0,
\]
and hence
\[
6a_{l+1} = - (a_0b_{l+1} + a_0a_1b_l + \ldots + a_0a_kb_0).
\]
We note that all the summands on the right hand side are divisible by \( a_0 \), therefore the left hand side also has this property. Hence \( a_0 \mid a_{l+1} \).

But we have \( a_k = \pm 1 \) and with no loss of generality we may take \( a_0 = 1 \) and then we have \( b_0 = 6 \).

Now if we repeat the above arguments for the coefficients of \( Q \), then we see that \( b_0 = 0 \) divides \( b_1, b_2, \ldots, b_{n-3} \) (we set, if necessary, \( b_i = 0 \) if \( i > n - k \)). Then we obtain a contradiction \( (b_{n-k} = \pm 1) \), unless \( n - k > n - 3 \). We consider two cases.

The case of \( k = 2 \). Then we arrive at
\[
a_0b_{n-k} + a_1b_{n-k-1} + \ldots + a_{n-k-1}b_1 + a_{n-k}b_0 = 0.
\]
a contradiction, because on the left hand side all the summands, except for the first one, are divisible by 6 and hence even, and the first summand is equal to ±1.

The case of \( k = 1 \). Then the problem reduces to finding an integer root of the polynomial \( W \); it is easy to check that if \( n \) is even there are no such roots; if \( n \) is odd, \( W(-1) = 0. \)

Therefore the answer to the problem is: \( n \) is odd.

4. We distribute \( n > 1 \) labelled balls among nine persons \( A, B, C, D, E, F, G, H, I \).

Determine in how many ways it is possible to distribute the balls under the condition that \( A \) gets the same number of balls as the persons \( B, C, D, E \) together.

Solution. Consider the polynomial

\[
(x + 2)^n - (x^2 + 4x + 4)^n = \binom{n}{2} \cdot 2^n.
\]

and suppose that we multiply out the brackets, obtaining \( 9^n \) summands. We show the one-to-one correspondence between the number of \( x^n \)'s and the number of the distributions we deal with in the problem.

Suppose we have such a distribution. If the \( k \)-th ball goes to \( A \), we pick \( x^2 \) from the \( k \)-th bracket. If it goes to \( B, C, D, E \), we pick the first, second, third or fourth \( 1 \), respectively, from the \( k \)-th bracket. If it goes to \( F, G, H, I \), then we take the first, second, third or fourth \( x \) from the \( k \)-th bracket. Now if we multiply the factors we have chosen, we see, that the result is equal to \( x^n \) if and only if \( A \) gets the same number of balls as \( B, C, D, E \) jointly.

Therefore, the number of the distributions we are interested in is equal to the coefficient at \( x^n \) in the polynomial \( (x + 2)^n \), that is, \( \binom{n}{2} \cdot 2^n. \)

5. Let \( ABCD \) be a given convex quadrilateral. Determine the locus of the points \( P \) lying inside the quadrilateral \( ABCD \) and satisfying

\[
[PAB] \cdot [PCD] = [PBC] \cdot [PDA],
\]

where \([XYZ]\) denotes the area of a triangle \( XYZ \).

Solution. If \( P \) lies on one of the diagonals \( AC \) or \( BD \), let’s say on \( AC \), then

\[
\frac{[PAB]}{[PBC]} = \frac{AP}{PC} = \frac{[PDA]}{[PCD]},
\]

which is the desired equality. We prove that no other point lying inside \( ABCD \) satisfies the conditions of the problem.

Denote by \( O \) the point of the intersection of the diagonals \( AC \) and \( BD \) and suppose that \( P \) lies inside the triangle \( ABO \). Let moreover \( BP \) and \( AC \) meet at \( Q \) and \( DP \) and \( AC \) meet at \( R \). Then

\[
\frac{[PAB]}{[PBC]} = \frac{AQ}{QC} \quad \text{and} \quad \frac{[PDA]}{[PCD]} = \frac{AR}{RC},
\]

which, since \( Q \neq R \), implies that the given equality cannot hold.

Therefore the desired locus of the points \( P \) consists of the diagonals \( AC \) and \( BD \).

6. Determine all pairs of integers \((x, y)\) satisfying the equation

\[
y(x + y) = x^3 - 7x^2 + 11x - 3.
\]

Solution. The considered equation is equivalent to

\[
(2y + x)^2 = 4x^3 - 27x^2 + 44x - 12 = (x - 2)((4x^2 - 19x + 6) = \quad = (x - 2)((x - 2)(4x - 11) - 16).
\]

The expression above must be a perfect square. Therefore we have either \( x = 2 \) (and \( y = 1 \)), or \( (x - 2) = k^2 \), where \( k \in \{-2, -1, 1, 2\} \) and \( s \in \mathbb{N} \); indeed, if for some prime \( p \) and nonnegative integer \( m \) the number \( p^{2m+1} \) divides \( (x - 2) \) but \( p^{2m+2} \) does not, then we have \( p | (x - 2)(4x - 11) - 16 \), so \( p | 16 \) and \( p = 2 \).

We will consider three cases separately.

The case of \( k = \pm 2 \). Then we have \( 4x^2 - 19x + 6 = \pm 2u^2 \) for some integer \( u \), or, equivalently,

\[
(8x - 19)^2 - 265 = \pm 32u^2,
\]

a contradiction modulo 5.

The case of \( k = 1 \). Then \( 4x^2 - 19x + 6 = u^2 \) for some integer \( u \), which leads to

\[
265 = (8x - 19)^2 - 16u^2 = (8x - 10 - 4u)(8x - 19 + 4u).
\]

We easily check that \( x = 6 \) is the only solution of this equation (we simply consider all possible decompositions: \( 265 = 1 \cdot 265 = 5 \cdot 53 = \ldots \), etc, and take into account the fact that \( x - 2 = u^2 \)). Therefore, we obtain two solutions of the original equation:

\(
(x, y) \in \{(6, 3), (6, -9)\}.
\)

The case of \( k = -1 \). As before, we have \( 4x^2 - 19x + 6 = -u^2 \), which is equivalent to

\[
265 = (8x - 19)^2 + (4u)^2
\]

and we check all possibilities with \( u \leq 4 \): for \( u = 0, 1, 2 \) there are no solutions. If \( u = 3 \), then we obtain \( 265 = 121 = 11^2 \), which leads to \( x = 1 \), which gives two solutions: \( (x, y) \in \{(1, 1), (1, -2)\} \). Finally, for \( u = 4 \) we arrive at \( (8x - 19)^2 = 9 = 3^2 \) and \( x = 2 \), which gives \( (x, y) = (2, -1) \).

Therefore, the set of solutions is as follows:

\[
\{(6, 3), (6, -9), (1, 1), (1, -2), (2, -1)\}.
\]